PAPER

Young children 'solve for *x*' using the Approximate Number System

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Abstract

The Approximate Number System (ANS) supports basic arithmetic computation in early childhood, but it is unclear whether the ANS also supports the more complex computations introduced later in formal education. 'Solving for x' in addend-unknown problems is notoriously difficult for children, who often struggle with these types of problems well into high school. Here we asked whether 4–6-year-old children could solve for an unknown addend using the ANS. We presented problems either symbolically, using Arabic numerals or verbal number words, or non-symbolically, using collections of objects while preventing verbal counting. Across five experiments, children failed to identify the value of the unknown addend when problems were presented symbolically, but succeeded when problems were presented non-symbolically. Our results suggest that, well before formal exposure to unknown-addend problems, children appear to 'solve for x' in an intuitive way, using the ANS.

Research highlights

- Children have difficulty solving problems with an unknown operand
- We showed children unknown-addend problems presented with Arabic digits, number words, or with collections of objects
- Children could only solve the problems that were presented non-symbolically
- Children can solve unknown-addend problems intuitively using the Approximate Number System prior to formal math experience

Introduction

The ability to represent and mentally manipulate exact quantities is unique to humans and depends on learning a verbally mediated system of number (Carey, 2009; Feigenson, Dehaene & Spelke, 2004). However, pre-verbal infants, non-verbal animals, children, and adults also have an Approximate Number System (ANS) that allows them to estimate quantities without language or instruction (Libertus & Brannon, 2009; Feigenson *et al.*, 2004; Dehaene, 1997). The ANS produces imprecise estimates of number, with the amount of noise in the numerical representations increasing with numerosity. As a result, numerical discrimination performance depends on the ratio between quantities rather than the quantities' absolute value. For example, 6-month-old infants can discriminate 8 dots from 16 (a 2:1 ratio), but not 8 dots from 12 (a 3:2 ratio; Xu & Spelke, 2000; Xu, 2003). Although ANS representations are inherently imprecise, this imprecision decreases over development (Halberda & Feigenson, 2008; Lipton & Spelke, 2004), and does not level off until well into adulthood (Halberda, Ly, Wilmer, Naiman & Germine, 2012).

The ANS supports a variety of number-relevant computations. For example, 11-month-old infants recognize ordered relations among dot arrays, dishabituating when sequences of arrays change from numerically ascending to descending or vice versa (Brannon, 2002). Preschoolers can indicate which of two approximate quantities is larger, even when the quantities are presented in different sensory modalities (Barth, LaMont, Lipton & Spelke, 2005). Children also can use the ANS to perform basic arithmetic operations without training. Nine-month-old infants who saw an array of dots become occluded, then saw a second array of dots added, looked

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longer when the occluder lifted to reveal an incorrect sum than a correct one (McCrink & Wynn, 2004). Six-month-old infants can detect a common ratio of blue to yellow shapes across arrays containing different absolute quantities, suggesting a rudimentary division computation (McCrink & Wynn, 2007). The ANS has even been shown to support approximate multiplication in 5- to 7-year-old children, prior to formal multiplication instruction (McCrink & Spelke, 2010).

As children get older and begin to master the symbolic system of exact number representation, the ANS continues to play an important role in their numerical thinking. Children can use the ANS to reason about arithmetic problems using number words and digits prior to receiving mathematical instruction (Gilmore, McCarthy & Spelke, 2007; Booth & Siegler, 2008; Barth, La Mont, Lipton, Dehaene, Kanwisher & Spelke, 2006). Further, individual differences in the precision of the ANS have been shown to correlate with differences in formal math ability as assessed by standardized math tests (Feigenson, Libertus & Halberda, 2013; Libertus, Feigenson & Halberda, 2013a, 2013b; Halberda, Mazzocco & Feigenson, 2008; Starr, Libertus & Brannon, in press). And math learning disability (dyscalculia) appears to be due at least in part to markedly poor ANS acuity (Mazzocco, Feigenson & Halberda, 2011; Piazza, Facoetti, Trussardi, Berteletti, Conte, Lucangeli, Dehaene & Zorzi, 2010).

These results suggest that the Approximate Number System plays an important role in numerical reasoning throughout the lifespan, both before and after formal instruction in mathematics. However, it appears that not all math abilities draw equally upon the ANS. Recent evidence suggests that whereas ANS precision predicts informal math abilities in young children (e.g. counting, informal calculation), no such relationship exists with formal math abilities (e.g. solving written math problems, understanding place values; Libertus *et al.*, 2013b). Therefore, it remains unclear the extent to which the ANS can be used for the more sophisticated mathematical reasoning that children increasingly encounter through formal schooling.

Algebraic reasoning is one type of mathematics that typically is not introduced until the late elementary or early middle school years (National Council of Teachers of Mathematics, 2000). Although there is much debate about how to differentiate algebra from arithmetic, one important link between the two is the ability to solve for an unknown value. Whereas solving for an unknown value that is the result of an operation (e.g. $5 + 7 = x$) is introduced early in elementary school and is more straightforwardly arithmetic, solving for an unknown addend, subtrahend, or minuend (e.g. $x + 5 = 12$) is introduced later and is significantly harder for children (e.g. Booth, 1988; Kieran, 1992; Filloy & Rojano, 1989; Koedinger, Alibali & Nathan, 2008; Riley & Greeno, 1988) and even for college-aged students (Tabachneck, Koedinger & Nathan, 1995). Herscovics and Linchevski (1994) have identified this as a 'cognitive gap' between arithmetic and algebra. According to Nathan and Koedinger (2000a), problems with an unknown addend or subtrahend 'tend to subvert simple modeling and direct calculation and often require algebraic methods'.

The finding that children struggle to solve problems with unknowns raises the question of whether intuitive approximate number representations can play a role in such computations. The difficulty that children evince with addend-unknown problems might reflect trouble making numerical inferences – for example, working backwards from a result to infer the identity of an unknown value. Alternatively, children may struggle to 'solve for *x*' because of the formal notation that is often used to present these problems (Nathan, 2012; Van Amerom, 2003; Herscovics & Linchevski, 1994). Previous findings suggest that children sometimes are more successful at solving math problems that are presented non-symbolically than when the same problems are presented using formal notation (Bisanz, Sherman, Rasmussen & Ho, 2005; Levine, Jordan & Huttenlocher, 1992). For example, 3-year-old children who are not yet proficient verbal counters demonstrate some understanding of mathematical inversion (e.g. $a + b - b = a$) when problems are presented with blocks (Sherman & Bisanz, 2007). And middle- and high-school students consistently perform better on word problems than on equivalent problems presented using equations and number symbols (Koendinger & Nathan, 2004). This suggests that children's mathematical intuitions sometimes precede their formal math abilities.

If representing exact quantities using formal notation impedes children's ability to solve for *x* in addend-unknown problems, then presenting the problems in a more intuitive way may reveal successful performance. The goal of the present studies was to determine whether 4- to 6-year-old children could spontaneously solve problems in which one of the addends was unknown, when the problems were presented non-symbolically. First, in Experiments 1 and 2, we confirmed that children were unable to solve for an unknown addend when the problems were presented using number symbols (Arabic numerals in Experiment 1; number words in Experiment 2). Then, in Experiments 3–5, we presented addend-unknown problems non-symbolically, using collections of objects, and asked whether children could identify the value of the unknown addend.

Experiment 1

Previous studies show that young children often have difficulty producing exact numerical values (LeCorre & Carey, 2007). Yet even 5-year-old children can succeed in symbolic arithmetic tasks when they are asked whether the solution to a problem is larger or smaller than a comparison number (rather than being asked to produce a precise numerical answer; Gilmore *et al.*, 2007). This shows that children's approximate number representations can allow them to compute over numerical symbols (even before children are proficient users of these symbols), but it remains unknown whether children also show this ability when faced with an unknown addend. Therefore, in Experiment 1 we asked whether 4- to 6-year-old children could solve addend-unknown problems involving Arabic numerals, when the task did not require them to generate an exact answer. We presented children with written symbolic problems in which there was an unknown addend, then asked them to choose which of two symbolically presented numbers should be in the unknown addend's place. We hypothesized that if children in Experiment 1 have an approximate sense of the value of the unknown addend, they should be successful in choosing between the correct value and an incorrect distractor.

Participants

Twenty-eight children (mean age: 63 months 9 days; range 50 months 1 day–83 months 20 days; 12 girls)

participated in the children's wing of a local science museum. Children received a sticker after participating.

Stimuli

Stimuli consisted of three orange cards measuring 31 cm × 9 cm. On the left side of each card was printed a mathematical equation containing Arabic numerals, with an unknown addend depicted by an empty box. Equations were printed in 72-point font in black ink. Table 1 shows the three equations used, here labeled A, B, and C (children did not see the equations labeled). Each card had a flap on the right side which, when opened, revealed a pair of printed number choices presented as Arabic numerals. Whether the correct answer was on the left or right side of the pair and whether the larger numerical choice appeared on the left or right side of the pair were counterbalanced across children.

Procedure

Children sat at a child-sized table across from the experimenter, and were shown the three cards, one at a time. The experimenter first said, 'We're going to play a numbers game.' She showed children the card containing Equation A and said, 'See this math problem?' She left the card visible for about 3 seconds, then flipped open the flap to reveal the pair of number choices. She pointed to the two choices, then to the empty box, and asked children which of the choices belonged in the empty box. Children were asked to point to one of the two numbers.

Table 1   *Quantities presented in Experiments 1–5*

| | | Demonstration Trials | | Test Trial |
	Presentation format	X	Y	Verbal Prompt & Quantity Choice
Exp. 1	Arabic numerals	*A.* **5 + □ = 17** *B.* **6 + □ = 42** *C.* **9 + □ = 13**		"Which goes in the box?" Child chooses between *A.* **12 & 4** *B.* **17 & 36** *C.* **4 & 12**
Exp. 2	Verbal	**5 + x = 17** **9 + x = 21** **6 + x = 18** *(x = 12)*		"Which cup is Gator's?" Child chooses between **12 & 4** *or* **12 & 24**
Exp. 3	Non-symbolic	**5 + x = 17** **9 + x = 21** **6 + x = 18** *(x = 12)*		"Which cup is Gator's?" Child chooses between **12 & 4** *or* **12 & 24**
Exp. 4	Non-symbolic	**5 + x = 41** **9 + x = 45** **6 + x = 42** *(x = 36)*		"Which cup is Gator's?" Child chooses between **36 & 17**
Exp. 5	Non-symbolic	**5 + x = 17** **9 + x = 21** **6 + x = 18** *(x = 12)*	**5 + y = 9** **9 + y = 13** **6 + y = 10** *(y = 4)*	"Whose cup is this?" Child shown either **12** *or* **4**

Regardless of children's choice, the experimenter said 'Great job!' This procedure was repeated for Equations B and C. Equations A, B, and C were presented in counterbalanced order across children.

Results

Children averaged 46% correct across the three equations (Table 2). For equation A, 14/28 children correctly chose the target number 12 (50%, binomial test $p = 1.0$), for equation B, 14/28 children correctly chose the target number 36 (50%, binomial test $p = 1.0$), and for equation C, 11/28 children correctly chose the target number 4 (36%, binomial test $p = .35$). There was no difference in children's performance on the three equations (chi^2 test $p = .73$), and girls' and boys' performance did not significantly differ (girls: 52% correct, boys: 41% correct, $t_{26} = 0.91$, $p = .37$, two-tailed). We also asked whether there was a relationship between children's age and their performance. We conducted a one-way ANOVA on children's ages with the number of problems children answered correctly (out of three possible) as a between-subjects variable. This analysis revealed no significant difference in age across performance levels ($F_{1,27} = 0.39$, $p = .26$).

Discussion

In Experiment 1, we found that 4- to 6-year-old children performed at chance when asked to choose the value of an unknown addend when problems were presented using Arabic numerals. This finding was not surprising, given the well-known difficulty that even older children have in solving written equations (e.g. Booth, 1988; Kieran, 1992; Herscovics & Linchevski, 1994; Filloy & Rojano, 1989; Koedinger *et al.*, 2008; Riley & Greeno, 1988). However, some evidence suggests that middle- and high-school students find verbal story problems, in which the addends and solution are embedded in a verbal narrative, easier to solve than equations presented using Arabic numerals (Koendinger & Nathan, 2004;

Table 2 *Percent of children choosing the correct value in Experiments 1–5*

	Presentation format	% correct	Binomial test *p*
Exp. 1	Arabic numerals	46%	*n.s.*
Exp. 2	Verbal	59%	*n.s.*
Exp. 3	Non-symbolic	71%	*0.03**
Exp. 4	Non-symbolic	79%	*0.007**
Exp. 5	Non-symbolic	75%	*0.02**

Koendinger *et al.*, 2008). Therefore, before investigating children's use of the Approximate Number System in addend-unknown problems (Experiments 3–5), we first asked whether they could solve such problems using verbal number symbols rather than written ones. In Experiment 2 children were tested with addend-unknown problems that were embedded within in a verbally presented story scenario.

Experiment 2

The verbal story scenario used in Experiment 2 was designed to match that in Experiments 3–5, in which collections of objects were shown but in which no number words were used. In this story scenario, we introduced children to a stuffed animal character, Gator, who had a 'magic cup' that always added a constant number of objects to an existing collection of objects. We showed children a box and verbally told children how many objects were in the box, without showing them the objects inside. Children then saw Gator's magic cup cover the box and were told that the cup 'added more' objects to the box. We then told children the ending number of objects, but did not tell children how many objects Gator's magic cup had added. Thus, the contents of Gator's magic cup acted as the unknown addend. Children saw three demonstrations of Gator's cup, and were reminded that the cup always added the same number of objects every time. We then asked whether children had inferred the approximate number of objects that had been in Gator's magic cup by asking them to choose between two options.

Participants

Thirty-two children (mean age: 64 months 3 days; range 49 months 25 days–83 months 12 days; 16 girls) participated at the science museum. Sixteen of these children had participated in Experiment 1 immediately before Experiment 2. All children received a sticker after participating.

Stimuli

The stimuli consisted of a small stuffed alligator toy and two 10 oz. white paper cups. On each of the three Demonstration Trials, a different 5 cm × 5 cm × 2 cm opaque paper box (one white, one yellow, and one blue) was used, each containing a different collection of small objects (buttons, pennies, or toy shoes). Children never saw the objects inside the boxes, but were told what the objects were.

Procedure

Demonstration Trials

On the first of three Demonstration Trials (Figure 1, top left panel), the experimenter showed children the stuffed animal, Gator, next to an inverted white paper cup. The experimenter pointed to the cup and said, 'This is Gator's magic cup. If I put a box of things in front of Gator, his magic cup will come and add *more* things to the box, and it's always going to add the *same number of things to the box every time*, no matter what.' The experimenter then said, 'Let's try it!' She placed a white box on the table, shook it so that children could hear that there were objects inside, and said, 'Do you hear Gator's

buttons in there? He has five. Five buttons!' She then said, 'Now watch carefully, here comes Gator's magic cup!' She covered the box completely with Gator's cup, shook the cup, and then lifted it. She then shook the box again and said, 'It worked! Gator has 17 now! Seventeen buttons!' Children never saw the contents of the box.

Children then saw two more Demonstration Trials involving different types of objects and different starting and ending quantities, in which Gator's magic cup always added the same number of objects to the existing collection (see Table 1). Before each Demonstration Trial, the experimenter said, 'Do you think Gator's magic cup will work on anything else? Let's try (pennies/shoes)!' In the second Demonstration Trial, Gator's magic cup transformed a blue box of 9 pennies into 21 pennies. In the third Demonstration Trial, it transformed a green box of 6 toy shoes into 18 shoes. Across all three Demonstration Trials, children were verbally told the starting and ending numbers, but did not see the object collections themselves.

Test Trial

On the single Test Trial, the experimenter told children she was going to try Gator's magic cup one more time. The experimenter then handed Gator to the child and pretended to look for Gator's magic cup under the table. She then placed two identical white paper cups upside-down on the table and said 'Oh no! I found two magic cups under the table, but I can't remember which one is Gator's! Can you help me?' She then pointed to each cup and told the child the quantity inside, saying, 'This cup has 12, and this cup has 4 [or 24]. Which cup is Gator's?' (Figure 1, bottom left panel). Whether the distractor cup contained 4 or 24, whether it appeared on the right or the left, and whether it was described first or second, were counterbalanced across children.

Quantities

In the three Demonstration Trials, Gator's cup always added 12 objects to starting quantities of 5 buttons, 9 pennies, and 6 shoes. The starting quantities varied across trials in order to highlight the fact that the number of objects added by the magic cup did not depend on the starting quantity. In the Test Trial, children were given a choice between the correct target quantity (12) and one of two distractor quantities: either one that was smaller than the target (4) or one that was larger (24). This allowed us to test whether children were responding based on an approximate representation of the unknown addend or whether they were using other strategies. For example, if children expected only that the
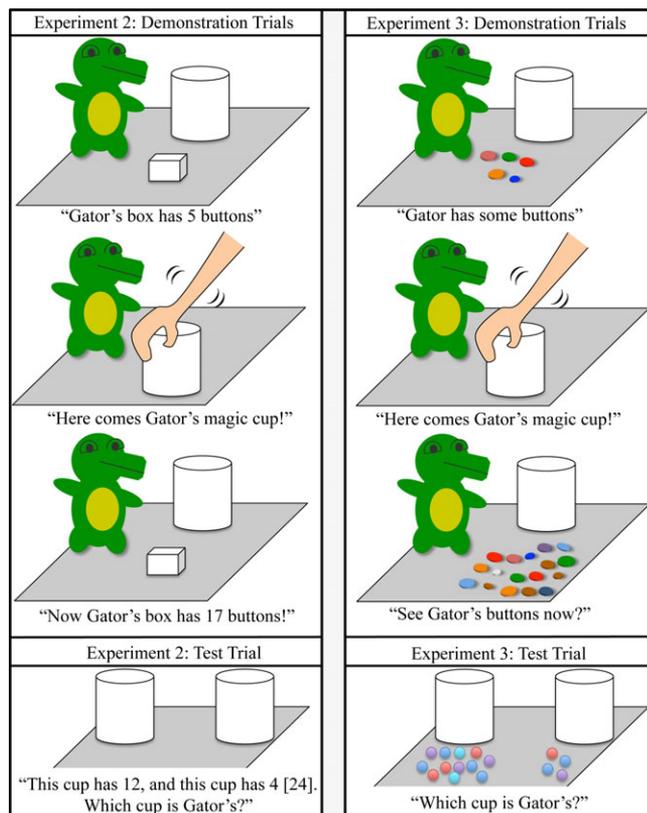

Figure 1   *Left panel: first of three Demonstration Trials in Experiment 2, in which children were told that the 'magic cup' added more buttons to an existing box of buttons, and the single Test Trial from Experiment 2, in which children were told the number of items in each of two cups and then chose which cup belonged to Gator. Right panel: first of three Demonstration Trials in Experiment 3, in which children saw the 'magic cup' add more buttons to an existing pile of buttons, and the single Test Trial in Experiment 3, in which children saw two quantities and then chose which cup belonged to Gator.*

unknown addend should be larger than the starting quantities, then they should succeed when choosing between 12 and 4, but fail when choosing between 12 and 24, because both 12 and 24 are larger than the starting quantities and therefore are ordinally consistent with an addition event. Further, the target and distractor quantities were chosen to be numerically close to numbers that children had heard during the Demonstration Trials so that the quantities would not be entirely novel (see Table 1), but were also chosen because they would be easily distinguishable by the ANS (Halberda & Feigenson, 2008).

Results

We found that 19/32 children chose the correct value of Gator's cup; this was not significantly different from chance (59%, binomial test $p = .38$, Table 2). We found no effect of the quantity in the distractor cup (11/17 children (65%) succeeded with 12 vs. 4; 7/15 (47%) succeeded with 12 vs. 24; Fisher's Exact Test $p = .48$, two-tailed), and no effect of gender (10/16 girls and 9/16 boys succeeded; Fisher's Exact Test $p = 1.0$, two-tailed). To assess whether there was a relationship between children's age and their performance, we analyzed whether there was a significant difference in the ages of children who succeeded versus those who did not using a one-way ANOVA with success as a factor. While children who succeeded were slightly older on average than children who failed, we found no significant difference in the ages of children who succeeded (mean age: 66 months 27 days; range: 52 months 9 days–83 months 12 days) versus those who failed (mean age: 60 months 0 days; range: 49 months 25 days–77 months 13 days) ($F_{1,30} = 3.38$, $p = .08$). We also asked whether children who had completed Experiment 1 prior to participating in Experiment 2 performed differently from children who had not. We found no difference in children's performance: 10/16 children who had participated in Experiment 1 succeeded (63%); of the remaining 16 children, 9 succeeded (56%; Fisher's exact test $p = 1.0$).

Discussion

Children in Experiment 2 failed to choose the value of an unknown addend, even when the problems were presented verbally in a story scenario with visual scaffolding (in the form of the boxes and the magic cup) instead of using Arabic numerals. Hence across Experiments 1 and 2, preschool- and early school-aged children failed to solve addend-unknown math problems when presented with number symbols that were either written (Experiment 1) or spoken (Experiment 2).

Children's failure with symbolically presented unknown addend problems raises the question of whether children of this age could succeed if they used the more intuitive, non-symbolic representations generated by the Approximate Number System. Therefore, in Experiment 3 we asked whether children could solve the same problems that had been presented in Experiment 2, but this time when the problems were presented entirely non-symbolically. Children were again introduced to Gator and his magic cup, but this time they watched the cup transform collections of visible objects, and did not encounter any digits or number words.

Experiment 3

Participants

Twenty-eight children (mean age: 64 months 5 days; range 48 months 21 days–81 months 26 days; 12 girls) participated, either in a university laboratory or at the science museum. Children tested in the laboratory were given a book or small toy for participating; children tested at the museum received a sticker.

Stimuli

The stimuli were identical to those in Experiment 2, except that no boxes were used. Instead, children saw piles of multi-colored buttons, pennies, or blue toy shoes. These three object collections were chosen to vary in size and shape in order to minimize children's attention to continuous dimensions (e.g. cumulative area) that are often confounded with number. In the single Test Trial, collections of small multi-colored pom-poms were used.

Procedure

Demonstration Trials

Children were introduced to Gator and his magic cup in the same manner as in Experiment 2. In the first Demonstration Trial, the experimenter placed a pile of 5 multi-colored buttons in front of Gator. She pointed to the buttons and said, 'See Gator's buttons?' The pile remained visible for approximately 5 seconds, then the experimenter said, 'Now watch carefully, here comes Gator's magic cup!' She covered the pile completely with Gator's cup (without showing children the cup's contents), shook the cup, and then lifted it to reveal 17 buttons (Figure 1, top right panel). The experimenter pointed to the pile and said, 'It worked! See Gator's buttons now?' Children had approximately 5 seconds to

view the change in the numerosity of the collection of buttons. All of the buttons and Gator's cup were then cleared from the table. Children then saw two more Demonstration Trials (see Table 1) with different object types and different starting and ending quantities, but with the same number of objects added each time, just as in Experiment 2. Before each trial, the experimenter said, 'Shall we try another one? Let's see if Gator's magic cup works on [pennies/shoes]!' In the second Demonstration Trial, children saw Gator's cup transform a pile of 9 pennies into 21 pennies, and in the third, children saw Gator's cup transform a pile of 6 toy shoes into 18 shoes. Children never heard any number words during the Demonstration Trials. The object collections were presented in tight clusters and were shown quickly so as to discourage children from serial counting. No child showed evidence of trying to count.

Test Trial

As in Experiment 2, on the single Test Trial, the experimenter told children she was going to try Gator's magic cup one more time. She gave Gator to the child, then placed two identical white paper cups upside-down on the table and said, 'Oh no! I found two magic cups under the table, but I can't remember which one is Gator's! Can you help me?' She then lifted the cups to reveal two different quantities of multi-colored pom-poms, placing the now-empty cups next to these piles. The experimenter then asked the child to put Gator with his magic cup (Figure 1, bottom right panel). As in the Demonstration Trials, no number words were used.

Quantities

The quantities used were identical to those in Experiment 2 (see Table 1). The quantity in the magic cup, 12, was visually discriminable from the average of the starting quantities (6.66) by a ratio of 2:1.

Results

We found that 20 out of 28 children (71%) successfully matched Gator to the target numerosity, significantly greater than chance (chance = 50%, binomial test $p = .03$, two tailed; Table 2). We found no effect of the quantity in the distractor cup: 10/14 children (71%) succeeded with 12 vs. 4; 10/14 (71%) succeeded with 12 vs. 24 (Fisher's Exact Test $p = 1.0$, two-tailed) and no effect of gender (10/12 girls and 10/16 boys succeeded; Fisher's Exact Test $p = .40$, two-tailed). There was also no difference in performance as a function of age; children who succeeded (mean age: 63 months 10 days,

range: 48 months 21 days–81 months 26 days) did not significantly differ in age from children who did not (mean age: 66 months 4 days; range: 49 months 6 days–82 months 20 days) (one-way ANOVA $F_{1,27} = 0.39$, $p = .54$).

Discussion

In Experiment 3, 4-to 6-year-old children watched non-symbolic events of the form $a + x = b$. We found that children successfully chose between the correct value of x and a distractor quantity, despite not being asked explicitly to track numbers of objects. Importantly, the quantities used in Experiment 3, in which children succeeded, were identical to those in Experiment 2, in which children failed; the only difference between the experiments was whether the quantities were presented visually or verbally.

Children in Experiment 3 were unlikely to have succeeded by counting. The objects were shown rapidly and were closely spaced so that it would have been difficult for children to serially count them, and all of the quantities presented were outside of the subitizing range (Gelman, 1977; Starkey & Cooper, 2011). Furthermore, since children in Experiment 2 apparently were unable to use number words to successfully infer the numerosity in the magic cup, it is unlikely that counting would have helped children in Experiment 3 infer the correct value of the unknown addend. Children also could not have succeeded by using the strategy of choosing the greater of the two quantities presented on the Test Trial. When given the choice between the target quantity 12 and the distractor quantity 24, children correctly chose 12, even though both 12 and 24 were ordinally consistent with an adding event (because both were greater than the numerosity of the object collection before it had been transformed by the magic cup).

Still, we wanted to rule out other possible explanations for children's success in Experiment 3. In particular, although the distractor quantities in Experiment 3 were chosen to minimize the impact of numerical novelty on children's choices, this design may have led children to rely on a simpler strategy of avoiding any quantity that was similar to the observed starting and ending quantities, rather than actually inferring the quantity in the magic cup. For example, if faced with a choice between 12 and 4, children could have chosen 12 because they recognized that 4 was similar to the starting quantities they had seen, and therefore was unlikely to be the target. If faced with a choice between 12 and 24, children could have chosen 12 because they recognized that 24 was similar to the ending quantities they had seen. To rule out this possibility, in Experiment 3 we presented

children with target and distractor quantities that were numerically in between the values of the starting and ending quantities, and asked whether children still could correctly choose the unknown addend.

Experiment 4

Participants

Twenty-four children (mean age: 63 months 5 days; range 48 months 14 days–81 months 26 days; 11 girls) participated at the science museum. Children received a sticker after their participation.

Stimuli

The stimuli were identical to those in Experiment 3.

Procedure

Demonstration Trials

Demonstration Trials were similar to Experiment 3, except that this time the magic cup always added 36 objects to the starting piles of 5 buttons, 9 pennies, and 6 shoes (Table 1).

Test Trial

The single Test Trial was structured as in Experiment 3. The experimenter gave Gator to the child, then placed two identical white paper cups upside-down on the table and said, 'Oh no! I found two magic cups under the table, but I can't remember which one is Gator's! Can you help me?' She lifted the cups to reveal two different quantities of multi-colored pom-poms, the target quantity 36 and a distractor quantity 17, placing the now-empty cups next to these piles. The experimenter then asked the child to put Gator with his magic cup. As in the Demonstration Trials, no number words were used.

Quantities

Throughout the three Demonstration Trials, children always saw Gator's magic cup add 36 objects to piles of 5 buttons, 9 pennies, and 6 shoes (Table 1). In the single Test Trial, children were presented with two identical cups, one containing the target quantity (36) and one containing the distractor quantity (17). These were chosen to fall between the average of the starting quantities (6.66) and the average of the ending quantities (42.66) seen across the three Demonstration Trials.

Results

We found that 19/24 children (79%) successfully chose the target numerosity (chance = 50%, binomial test $p = .007$, two-tailed, Table 2). There was no effect of gender: 8/11 girls and 11/13 boys succeeded (Fisher's Exact Test $p = .63$, two-tailed). Again, we found no significant difference in the ages of children who succeeded (mean age: 64 months 2 days, range: 48 months 14 days–81 months 26 days) versus those who failed (mean age: 59 months 26 days; range: 48 months 14 days–71 months 0 days) (one-way ANOVA $F_{1,22} = 0.87$, $p = .36$).

Discussion

Experiment 4 replicated the results of Experiment 3: 4- to 6-year-old children were able to infer the approximate value of an unknown addend. Importantly, Experiment 4 showed that children succeeded even when prevented from using a simpler strategy of choosing a target value that was outside the range of the starting and ending quantities.

The results of Experiments 1–4 suggest that children have some proficiency in reasoning about addend-unknown problems before they have amassed much, if any, education in mathematics, and before they can solve these types of problems using symbolic representations. However, our results thus far leave open the question of whether children mentally *bound* a particular quantity to a specific variable. It is possible children in Experiments 3 and 4 simply had an unbound, diffuse sense of the 'missing quantity' gleaned from a general representation of the difference between the starting and ending numbers. We tested this in Experiment 5 by presenting children with two hidden values instead of just one. Children were introduced to two characters, Gator and Cheetah, who each had magic cups that added different amounts. If children in Experiments 3 and 4 were responding on the basis of a global perceptual change in number across trials, then children in Experiment 5 should be unable to identify which cup belonged to which character. However, if children succeed in Experiment 5, it would suggest that they were indeed reasoning about the hidden quantities and binding those quantities to specific unknown addends.

Experiment 5

Participants

Twenty-four children (mean age: 64 months 2 days, range 49 months 22 days–83 months 2 days; 16 girls)

participated at the science museum. Children received a sticker after participating.

Stimuli

Stimuli were identical to Experiments 3 and 4, except that a second stuffed animal, Cheetah, and an additional 10 oz. white paper cup were also used.

Procedure

Demonstration Trials

Children were introduced to Gator and his magic cup in the same manner as in Experiments 3 and 4. On the first Demonstration Trial they watched as Gator's magic cup transformed a pile of 5 buttons into 17 buttons. Next, children were introduced to Cheetah and were told that Cheetah also had a magic cup (which looked identical to Gator's), but that his magic cup always added a *different* number of objects from Gator's magic cup. The experimenter placed Cheetah's cup upside-down on the table. She put a pile of 5 buttons in front of Cheetah, covered them completely with Cheetah's magic cup, shook it, and then lifted the cup to reveal 9 buttons.

Children then saw two more Demonstration Trials (Table 1), in which Gator's magic cup and Cheetah's magic cup were seen to transform starting piles of objects that were always of the same type and had the same starting numerosity. In the second Demonstration Trial, children saw Gator's cup transform a pile of 9 pennies into 21 pennies, and then saw Cheetah's cup transform a pile of 9 pennies into 13 pennies. In the third Demonstration Trial, children saw Gator's cup transform a pile of 6 toy shoes into 18 shoes, and then saw Cheetah's cup transform a pile of 6 shoes into 10 shoes. Whether Gator or Cheetah was presented first within each trial, and whether Gator's cup or Cheetah's cup added 12 objects (versus 4 objects) were counterbalanced across children.

Test Trial

On the single Test Trial, the experimenter told children that she was going to try Gator's and Cheetah's magic cups one more time. She gave children both Gator and Cheetah while she pretended to look for their cups under the table. She then showed children just one magic cup and said, 'Uh-oh! I found a magic cup under the table but I don't know if it is Gator's cup or Cheetah's cup. Can you help me?' The experimenter lifted the cup to reveal either 4 or 12 pom-poms (counterbalanced across children) and said, 'Whose cup is this?'

Quantities

The starting quantities used were identical to Experiment 3 (see Table 1). The second unknown addend, 4, was chosen to be discriminable from the starting quantities by a ratio of at least 2:3 (on average, across the three different starting quantities), a ratio that is known to be discriminable even by 9-month-old infants (Xu & Spelke, 2000). This helped to ensure that children would be able to detect that the cup had added a quantity to the starting number.

Results

We found that 18 out of 24 children (75%) successfully matched the correct character to the target numerosity (chance = 50%, binomial test $p = .02$; Table 2). More girls than boys succeeded (14/16 girls versus 4/8 boys), although this difference was not significant (Fisher's exact test $p = .13$). There was no effect of the quantity in the test cup (9/11 children succeeded when the cup revealed 4 objects; 9/13 children succeeded when the cup contained 12 objects; Fisher's exact test $p = .65$). There was also no difference in the ages of children who succeeded (mean age: 63 months 20 days; range: 50 months 24 days–77 months 15 days) versus those who did not (mean age: 65 months 4 days; range: 49 months 22 days–83 months 2 days) (one-way ANOVA $F_{1,22} = 0.09$, $p = .76$).

Discussion

Children's success in Experiment 5 suggests that they had inferred the values of both hidden quantities (since they could not have known in advance which quantity would be queried by the experimenter in the test trial), and that they had bound each hidden quantity to a specific unknown. Children could not have succeeded by using a simple strategy of remembering which character ended up with an ending quantity that was 'more' or 'less' than the other: since children were only shown one quantity in the final test trial, they had to match that quantity with a specific character. Thus, 4- to 6-year-old children successfully inferred the values of two unknown addends.

General discussion

Previous work has shown that the Approximate Number System plays an important role in children's informal math abilities, supporting approximate arithmetic computation from infancy onward (e.g. McCrink & Wynn,

2004, 2007; Barth *et al.*, 2005). Here, in a series of five experiments, we asked whether the ANS also plays a role in children's reasoning in addend-unknown problems of the form $a + x = b$, long before children are formally introduced to these types of problems in school.

In Experiment 1, we gave 4- to 6-year-old children addend-unknown problems presented in formal symbolic notation using Arabic numerals, and asked whether they could choose which of two numbers was the value of the unknown addend. We found that children chose at chance, confirming that symbolically presented addend-unknown problems are difficult for children of this age. In Experiment 2, we asked whether children could solve addend-unknown problems involving verbal number symbols rather than written ones. We found that when the numbers were presented verbally in a story scenario, children again performed at chance. Then, in Experiments 3–5, we gave children addend-unknown problems in which all of the quantities were presented non-symbolically using piles of objects that were tightly clustered and were shown briefly – these features were selected to discourage verbal counting and encourage the use of the Approximate Number System. This time we found that children successfully inferred the value of the unknown in problems with a single unknown addend (Experiments 3 and 4) and with two unknown addends (Experiment 5).

Herscovitz and Linchevski (1994) noted the presence of a 'cognitive gap' between solving problems with an unknown solution and solving problems with an unknown operand, characterized by children's inability to 'operate spontaneously with or on the unknown' in unknown-operand problems. The present results suggest that this 'cognitive gap' may diminish when problems are presented in a way that taps children's intuitive number sense. Our work thus extends previous findings that children's mathematical intuitions sometimes precede their formal abilities, and suggest that children's well-known difficulty in mastering algebra may be influenced by their difficulty manipulating symbols using formal rules (Nathan, 2012; Van Amerom, 2003; Nathan & Koendinger, 2000a, 2000b; Nathan, Long & Alibali, 2002).

Children's success with problems that were presented entirely non-symbolically using collections of three-dimensional objects, and their failure with problems presented symbolically, raises the question of how numerical format affects algebraic reasoning. Previous research with high-school students who are learning algebra has shown that story problems involving unknown addends are easier than problems presented using formal notation (Koendinger & Nathan, 2004; Koedinger et al., 2008). Indeed, even young children have shown at least partial success when presented with

verbal number symbols (i.e. number words) rather than written ones. For example, when verbally presented with math problems in which an initial amount was unknown (e.g. 'There were some dogs at a party, then 3 more dogs came, so there were 5 dogs at the party. How many were at the party first?'), 5- and 6-year-old children, but not 4-year-old children, responded with answers that were not exactly right, but were in the right ordinal direction. That is, children gave answers that were smaller than the final number when the problem involved addition, and larger than the final number when the problem involved subtraction (Sophian & McCorgray, 1994; see also Carraher, Schliemann & Schwartz, 2008). Thus although children were unable to infer the value of the unknown, they did have a limited sense of its magnitude.

However, our results suggest that when children are able to rely on the ANS, they can do more than simply recognize the ordinal direction in such numerical transformations. When we asked 4- to 6-year-old children to choose between the target quantity and a distractor quantity, they chose successfully even when both options were in the right ordinal direction. Further, children as young as 4 years old succeeded in all of our non-symbolic tasks, whereas 4-year-olds were unable to produce directional responses when given word problems in the study by Sophian and McCorgray (1994). This suggests that presenting information in a way that promotes the use of ANS representations may scaffold children's ability to reason about unknown quantities.

Our results also shed further light on arithmetic processing by the ANS. Previous research has shown that even young infants can use the ANS to add and subtract collections of items in an approximate way (e.g. McCrink & Wynn, 2004). This kind of non-symbolic arithmetic can be accomplished by simply incrementing or decrementing a single accumulator each time an array is transformed. In other words, a single running total can be represented, with the current magnitude corresponding to the sum or the difference of two observed quantities. Our task, in contrast, could not be accomplished by incrementing a single accumulator. In order to infer the quantity of the unknown addend, children had to maintain representations of two separate quantities, the starting quantity and the ending quantity, and later combine these (for example, by subtracting the approximate starting quantity from the approximate ending quantity) to determine the approximate value of the unknown addend.

In this sense, solving for an unknown addend using the ANS may impose greater demands on working memory than solving an arithmetic problem using the ANS, since two or more quantities must be maintained in memory and combined. Previous research has shown that both

adults (Feigenson, 2008; Halberda, Sires & Feigenson, 2006) and infants (Zosh, Halberda & Feigenson, 2011) can maintain up to three numerical approximations in working memory concurrently, but little is known about how individual differences in working memory may interact with non-symbolic numerical processing using the ANS. Given the large amount of work demonstrating a relationship between symbolic math and working memory (e.g. Ashcraft & Kirk, 2001; Bull & Scerif, 2001; McLean & Hitch, 1999), this is an important direction for future investigation.

Our results also highlight a number of other questions for future research. First, how sophisticated is children's ability to intuitively infer the values of unknown operands? When presented with equations that require multiple operations (e.g. $5 + x = 29 - y$), can children still draw on ANS representations to solve non-symbolically presented problems? Second, how are the operations that are defined over ANS representations related to the kinds of formal operations that are required to solve symbolic addend-unknown problems, if they are related at all? For example, can presenting children with non-symbolic addend-unknown problems scaffold children's ability to solve these problems when presented symbolically? Answers to these questions will help to elucidate the link between the early abilities revealed by the present experiments, and the formal algebraic skills that children acquire in the years to come.

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