Systems Engineering PhD Qualifying Exam

WRITTEN EXAM: Tuesday, May 21st 2024
9:00AM-1:00PM, 15 Saint Mary's Street, Room 105

- **NO ELECTRONIC DEVICES** (smartphone, iPad, smartwatch) permitted
- Calculators and a ruler are allowed.
- CLOSED BOOK. Only the notes indicated below will be allowed.

INSTRUCTIONS:

1) Write your **EXAM NUMBER** on every sheet of paper

2) Write clearly and legibly as the exam may be scanned to faculty for grading.

3) **Answer 3 out of 5 questions** completely from the five sections below:

Section I: Dynamic Systems Theory (SE 501, Baillieul)
  - CLOSED BOOK, NO NOTES

Section II: Discrte Stochastic Processes (SE 714, Perkins)
  - CLOSED BOOK, NO NOTES

Section III: Optimization (SE 674, Paschalidis)
  - CLOSED BOOK, 2 8.5 x 11 sheets (4 pages) of handwritten notes

Section IV: Dynamic Programming and Stochastic Control (SE 710, Caramanis)
  - CLOSED BOOK, NO NOTES

Section V: Kinetic Processes in Materials (SE 762, Wang)
  - CLOSED BOOK, NO NOTES

*AM refreshments and grab n go lunch boxes served

ORAL EXAM: Thursday, May 23rd 2024
9:00AM-6:00PM, 15 St. Mary's Street, Room 121

Time Slot	Student	Committee
9AM – 10AM		
10AM - 11AM		
11AM – 12PM		
12PM-1PM		
1PM-2PM		
2PM-3PM		
3PM-4PM		
4PM-5PM		
5PM-6PM		

Systems Engineering Control Qualifying Exam — May 2024

1. (a) The matrix exponential e^{Mt} where

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

has the form $A + B \sinh t + C \cosh t$. Find A, B, C.

(b) Write a closed-form expression for the matrix exponential e^{Nt} where

$$N = \begin{pmatrix} \lambda & 1 & 0 \\ 1 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

2. Consider the system $\dddot{x} = u, \quad y = x$.

(a) Write the system in (first order) state-space form.

(b) Is the system you have written *controllable*? *Observable*?

(c) Design a state feedback control law such that the closed loop system has all poles at -1.

(d) Design a full-state observer with poles at $-1, -2, -3$.

3. (a) State in words the definitiion of *controllability*.

(b) Given a constant coefficient linear system $\dot{x} = Ax + Bu$ with $x \in \mathbb{R}^n$,the following are known to be equivalent:

(i) The system is controllable.

(ii) $rank\ (B, AB, \ldots, A^{n-1}B) = n$.

(iii) $\lambda \in \mathbb{C},\ p^T A = \lambda p^T,\ p^T B = 0 \Rightarrow p = 0$.

(iv) $rank\ [\lambda I - A,\ B] = n\ \forall \lambda \in \mathbb{C}$.

Prove (ii) $\Rightarrow$ (iii)& (iv).

4. (a) For the finite dimensional linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

Please turn over→

where $A(t)$ and $B(t)$ are $n \times n$ and $n \times m$ matrices, consider the problem of steering the system from $x_0 \in \mathbb{R}^n$ to $x_1 \in \mathbb{R}^n$ in $T > 0$ units of time so as to minimize

$$\eta = \int_0^T \|u(t)\|^2 \, dt.$$

(a) Prove that the optimal value of η is

$$\eta_0 = [x_0 - \Phi(0,T)x_1]^{\mathrm{T}} \, W(0,T)^{-1} \, [x_0 - \Phi(0,T)x_1] \, .$$

where $W(0,T)$ is the *controllability Grammian*:

$$W(0,T) = \int_0^T \Phi(0,t)B(t)B(t)^T \Phi(0,t)^T \, dt.$$

(b) Consider two finite dimensional linear systems in $\mathbb{R}^n$:

$$\dot{x} = Ax + b_1 u, \text{ and}$$
$$\dot{x} = Ax + b_2 u,$$

where A is an $n \times n$ constant matrix, and b_1, b_2 are non-zero n-vectors such that the (A, b_1) system is controllable, but the (A, b_2) system is not controllable. Let $B = (b_1 \vdots b_2)$ be the $n \times 2$ matrix whose columns are the vectors b_1 and b_2. Prove that the two-input system

$$\dot{x} = Ax + Bu \; (= Ax + b_1 u_1 + b_2 u_2)$$

is controllable.

(c) Let x_0, x_1 be arbitrary points in $\mathbb{R}^n$. Compare the costs of steering each of the three systems in part (b) from x_0 to x_1 in T-units of time. Are the costs all defined, and among them, which has the smallest value?

Systems Ph.D. Qualifying Examination
May 2024
Section II: Discrete Stochastic Processes

Problem 1. Consider successive flips of a fair (i.e., unbiased) coin. Determine the expected total number of flips until the pattern HHTHHTH occurs for the first time.

Problem 2. Consider the Markov chain given by the one-step transition probability matrix

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0.25 & 0.25 & 0 & 0.25 & 0.25 \end{bmatrix}$$

with state space $S = \{0, 1, 2, 3, 4\}$.

a. Determine all communication classes of this Markov chain and, for each class, specify if it is positive-recurrent, null-recurrent, or transient. For each recurrent class, determine its period.

b. Describe all stationary distributions of the chain.

c. Determine $E[T_{3,3}]$ and $E[T_{4,4}]$, where $T_{i,j}$ is the first passage time from state i to state j.

d. Determine all eigenvalues of P and their multiplicities.

e. Determine $\lim_{n\to\infty} P^{(2n+1)}$

f. Given the initial state $\pi(0) = [.2 .2 .2 .2 .2]$ (that is, $\pi_j(0) = .2$ for $j = 0, 1, 2, 3, 4$), will the limiting probability vector $\lim_{n\to\infty} \pi(n)$ exist and be unique? If so, determine this vector. If not, explain why.

Problem 3. Recall that N is **stopping time** with respect to a sequence of random variables $\{X_n : n \geq 1\}$ if the event $\{N \geq n\}$ is independent of $\{X_n, X_{n+1}, X_{n+2} \ldots\}$. Consider a renewal process defined by the inter-renewal times $\{X_n : n \geq 1\}$. Let S_j be the time of the j^{th} renewal, $N(t)$ be the number of renewals in $(0, t]$, and $m(t) = E[N(t)]$ be the renewal function. Define the age at time t as $A(t) = t - S_{N(t)}$ and the residual life at time t as $Y(t) = S_{N(t)+1} - t$. Assume $E[X_1] < \infty$, $E[X_1^2] < \infty$ and the renewal process is not lattice.

a. Write $\lim_{t\to\infty} E[Y(t)]$ and $\lim_{t\to\infty} E[A(t)]$ in terms of $E[X_1]$ and $E[X_1^2]$. [You do not need to derive these equations.]

b. Using either the definition of $A(t)$ or the definition of $Y(t)$ determine

$$\lim_{t\to\infty} \left(m(t) - \frac{t}{E[X_1]} \right)$$

Division of Systems Engineering
Ph.D. Qualifying Exam

Boston University
May 2024

Area Qualifying Exam in Optimization

Due: Tuesday, May 21, when instructed by SE proctors.

Last Name	First Name	Student ID #

Honor Code: I have not received nor given any unauthorized aid during this exam in accordance with the indicated rules communicated by the SE Division. I further certify that all work is entirely my own and does not violate the Boston University Academic Conduct Code.

Signature: ____________________

- The exam is closed book; you can only use your cheat sheets (4 pages, 8.5x11 of your own notes).
- Calculators and computing devices would not be needed. Communication devices and access to the Internet are *not* permitted.
- It goes without saying, but any form of collaboration or help from others will not be tolerated and, if detected, it will be taken very seriously and will have consequences.
- All work you want graded must go in this exam booklet.

*** Good Luck!!! ***

Problem	Points earned	out of
Problem 1		30
Problem 2		70
Total		100

Problem 1 $5 \times 6 = 30$ *points*

For each one of the following statements please state whether they are true or false, **with** a detailed justification. LP refers to a linear programming problem. Grading will be done as follows:
Correct answer with correct justification: 6 points,
Correct answer with missing or wrong justification: 4 points,
No answer: 1 point,
Wrong answer: 0 points.

1. Consider the LP $\min_x\{\mathbf{c}'\mathbf{x} \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ and assume it is feasible. It has finite cost if and only if $\mathbf{c}$ can be written as a nonnegative combination of the rows of $\mathbf{A}$.

2. A problem in which a piecewise linear and concave function has to be minimized subject to linear constraints can be transformed into a linear programming problem.

3. Consider a positive definite (hence, symmetric) matrix $\mathbf{X}$. Then its maximum eigenvalue $\lambda_{\max}(\mathbf{X})$ is convex in $\mathbf{X}$.

4. LPs can be solved in polynomial time and the simplex method is an algorithm which can be used to that end.

5. It is impossible for both the problems in a primal-dual pair of LPs to have unbounded objectives.

Problem 2 $25 + 20 + 25 = 70$ *points*

Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{b}$ be a vector in $\mathbb{R}^m$. Consider the problem of minimizing $\|\mathbf{Ax} - \mathbf{b}\|_\infty$ over $\mathbf{x} \in \mathbb{R}^n$ and let v be the optimal value. Here, $\| \cdot \|_\infty$ is the vector norm defined by $\|\mathbf{y}\|_\infty = \max_{i=1,\ldots,m} |y_i|$ for $\mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$.

(a) Formulate this problem as a linear programming problem.

(b) Let $\mathbf{p} \in \mathbb{R}^m$ such that $\sum_{i=1}^m |p_i| = 1$ and $\mathbf{p}'\mathbf{A} = \mathbf{0}'$. Show that $\mathbf{p}'\mathbf{b} \leq v$.
Hint: $\|\mathbf{Ax} - \mathbf{b}\|_\infty = \|\mathbf{Ax} - \mathbf{b}\|_\infty \sum_{i=1}^m |p_i|$.

(c) To obtain the best possible bound in (b), we form the linear programming problem

$$\begin{array}{rl} \max & \mathbf{p}'\mathbf{b} \\ \text{s.t.} & \mathbf{p}'\mathbf{A} = \mathbf{0}', \\ & \sum_{i=1}^{m} |p_i| \leq 1. \end{array} \tag{1}$$

Show that the optimal cost of this problem is equal to v.
Hint: Use duality ...

Qualifying Examination in Dynamic Programming (SE710). *Closed book and notes*!

Consider a stochastic shortest path problem defined on a network connecting n nodes containing the terminal node n^*. The node that the system resides at time t is the observable state of the system at initial time t, $x(t)=i$, i in $\{1,2,3,\ldots,n^*,..N\}$. There are M discrete controls u^m , with m in $\{1,2,3,\ldots,M\}$ which can be applied at will after state i is observed and guide the state from $x(t)=i$ to $x(t+1)=j$ with unknown probabilities $p(i,j,u^m)$, which, however, are known to result in a communicating state transition probability controlled system. The period cost, $g_t(i,j,u^m)$, is also observable.

Consider a Q function with input in the state-control space,

$Q(i,u^m) \equiv \Sigma j\ p(i,j,u^m)\{\ g_t(i,j,u^m)+J_{\mu(j)}(j)\}$

where $\mu(j)$ is a given policy for each j used from time $t+1$ till the system arrives at the terminal node. You can assume that the controlled system associated with policy $\mu(j)$ is uni-chain and that the expected time to arrive at the terminal node is finite.

Devise a "*Q* learning Algorithm" that can be used to asymptotically derive $J_{\mu^*(j)}(j)$ where $\mu^*(j)$ is the optimal policy for all j in $\{1,2,3,\ldots,n^*,..N\}$. More specifically:

1. Design the algorithm to be driven by input consisting of:
   #the observed initial state i
   #the initially selected control, u^m
   #the observed next state j and the observed period cost $g_t(i,j,u^m)$
   #the next control selected, $u^{m'}$
   #the observed next state j' and period cost $g_{t+1}(j,j',u')$
   #the observed next states and period costs that follow, as well as the selected controls that follow.
2. Describe rules, if any, on selecting controls u^m, $u^{m'}$, $u^{m''}$,... at each time period and on updating Q (Hint: Robbins-Monroe) to guarantee convergence to $Q^*(i,u^m) \approx g_T(i,j,u^m)+J_{\mu^*(j)}(j)$ for large T.
3. Discuss how $\mu^*(j)$ is derived/related to $Q^*(i,u^m)$.

SE Qualifying Exam
Nonlinear Systems and Control
May 21, 2024

1. Give a precise mathematical definition of the following terms:

   (a) Uniformly asymptotic stability of an equilibrium point x_0 of $\dot{x}(t) = f(t, x(t))$.

   (b) Positive definiteness of a continuous function $W : [t_0, \infty] \times R^n \to R$.

   (c) Positive limit set of a bounded solution $x(t, x_0, t_0)$.

2. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1 f(x_1, x_2) \\ \dot{x}_2 &= -x_1 - x_2 f(x_1, x_2) \end{aligned}$$

   where $f(x_1, x_2)$ is a convergent power series with $f(0,0) = 0$. Determine the stability of $(0,0)$ for the following cases:

   (a) $f(x_1, x_2) \geq 0, \quad \forall x \in B_r(0)$

   (b) $f(x_1, x_2) > 0. \quad \forall x \in B_r(0)$

   (c) $f(x_1, x_2) < 0, \quad \forall x \in B_r(0)$

3. Consider the following nonlinear control system

$$\begin{aligned} \dot{x}_1 &= (-1-\alpha)x_1 - 2x_2 + (1+\alpha)u - ux_1(1-\alpha) \\ \dot{x}_2 &= (1-\alpha)x_1 + (1-\alpha^2)u - ux_2(1-\alpha) \end{aligned}$$

   where α is a real number. Determine for which values of α there exists a continuously differential feedback control $u = k(x)$, $k(0) = 0$ such that 0 is an asymptotically stable equilibrium point of the resulting closed loop system.