SYSTEMS ENGINEERING
PHD QUALIFYING EXAM
May 22, 2023, 9:00AM to 1:00PM, 110 Cummington Mall, room 245

CLOSED BOOK,
NO CHEAT SHEETS unless noted
BASIC SCIENTIFIC CALCULATOR PERMITTED
ALL EXAM MATERIALS STAY IN THE EXAM ROOM

GENERAL INSTRUCTIONS:

1) **Please write on every sheet:**
   a. Your Exam Number
   b. The page numbers (example: Page 1 of 4)

2) **Only write on 1 side.**
   Exams may be scanned and emailed to the faculty for grading. If using pencil, make sure it is dark.

COMPLETE THE REQUIRED SECTIONS AS BELOW:

The exam consists of **three topical sections**. You must complete **three** of the following **five sections**:

A. Dynamic Systems Theory (SE/EC/ME 501)
B. Discrete Stochastic Processes (EK500 and SE/ME 714)
C. Optimization (SE/EC 524)
D. Dynamic Programming and Stochastic Control (SE/EC/ME 710)
E. Nonlinear Systems and Control (SE/ME 762)

ORAL EXAM TBD: **Wednesday, May 24 - Friday, May 26, 2023**
9:00AM-4:00PM, 15 St. Mary's Street,
Time/Location TBA (~1 hour per student)

SE Linear Systems Qualifying Exam

May 22, 2023

1. In this problem, all questions concern a state-space control system of the form

$$\dot{x}(t) = Ax(t) + bu(t),$$

where A is an $n \times n$ matrix with constant entries and b is an $n \times 1$ matrix, also with constant entries.

(a) What does it mean for the system to be *controllable*?

(b) Assuming the system is controllable, show that there is a choice of coordinates in terms of which the matrices A and b have the respective forms

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

where the a_i's are coefficients of the characteristic polynomial of A, $p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$.

(c) Using the results of part (b), show that if the system is controllable,one can use state feedback $u = kx$ to assign values to all eigenvalues of the closed-loop system $\dot{x} = (A + bk)x$.

(d) Show that changing the system by means of state feedback cannot alter whether the system is controllable. This is to say that for any feedback $u = kx$, the closed-loop system

$$\dot{x} = (A + bk)x + bv$$

is controllable $\Leftrightarrow$ the system

$$\dot{x} = Ax + bu$$

is controllable.

(e) Write the second-order system $\ddot{x} = u$ in first order form, and find a state feedback law that places the closed-loop eigenvalues at -1 and -2.

(f) For the system of part (e), what is the state feedback law that minimizes the infinite horizon performance criterion

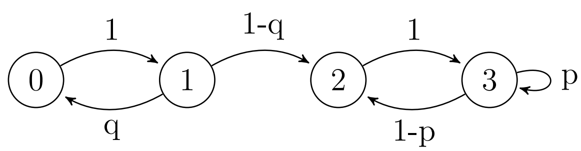
$$\int_0^\infty \|x\|^2 + u^2 \, dt \ ?$$

Systems Ph.D. Qualifying Examination
May 2023
B.2. Discrete Stochastic Processes

Problem 1. Suppose that $\{N(t) : t \geq 0\}$ is a homogeneous Poisson process with rate λ. For $n \geq k$ and $s, t \geq 0$, determine

$$P\left(N(t) = k | N(s+t) = n\right)$$

Problem 2. Consider the following discrete-time Markov chain with $0 < p, q < 1$:



a. Determine all communication classes of this Markov chain and, for each class, specify if it is positive-recurrent, null-recurrent, or transient. For each recurrent class, determine its period.

b. Determine all stationary distributions of the chain.

c. Determine $\lim_{n\to\infty} P^{(2n+1)}$.

d. Determine $E[T_{2,2}]$ and $E[T_{3,3}]$, where $T_{i,j}$ is the first passage time from state i to state j.

e. Determine $m_{1,0}$ and $m_{0,3}$, where $m_{i,j}$ is the expected total amount of time spent in state j starting from state i.

Problem 3. Consider a renewal process $\{N(t) : t \geq 0\}$. Define $Y(t) = S_{N(t)+1} - t$ to be the residual renewal time (or residual life) at time $t \geq 0$. Assume the inter-renewal times are uniformly distributed on the interval $[0, 10]$, i.e., $X_n \sim U[0, 10]$ for $n = 1, 2, \ldots$

a. Determine
$$\lim_{t\to\infty} \frac{1}{t} \int_0^t Y^2(u)du.$$

b. For $x > 0$, determine
$$\lim_{t\to\infty} \frac{1}{t} \int_0^t P\left(Y(u) > x\right) du.$$

c. Suppose this renewal process is observed at random in the extremely distant future. Let Y be time interval from when the process is observed until the next renewal occurs. Determine the probability density function for Y.

Division of Systems Engineering
Ph.D. Qualifying Exam

Boston University
May 2023

Area Qualifying Exam in Optimization

Due: Monday, May 22, when instructed by SE proctors.

Last Name	First Name	Student ID #

Honor Code: I have not received nor given any unauthorized aid during this exam in accordance with the indicated rules communicated by the SE Division. I further certify that all work is entirely my own and does not violate the Boston University Academic Conduct Code.

Signature: ______________________________

- The exam is closed book; you can only use your cheat sheets (6 pages, 8.5x11 of your own notes).
- Calculators and computing devices would not be needed. Communication devices and access to the Internet are *not* permitted.
- It goes without saying, but any form of collaboration or help from others will not be tolerated and, if detected, it will be taken very seriously and will have consequences.
- All work you want graded must go in this exam booklet.

*** GOOD LUCK!!! ***

Problem	Points earned	out of
Problem 1		40
Problem 2		60
Total		100

Problem 1 $5 \times 8 = 40$ *points*

For each one of the following statements please state whether they are true or false, **with** a detailed justification (no rigorous proof is required but you are welcome to provide one). LP refers to a linear programming problem. Grading will be done as follows:
Correct answer with correct justification: 5 points,
Correct answer with missing or wrong justification: 2 points,
No answer: 1 point,
Wrong answer: 0 points.

1. Consider the problem of minimizing $\max\{\mathbf{c}'\mathbf{x}, \mathbf{d}'\mathbf{x}\}$ over some polyhedron $\mathcal{P}$. If the problem has an optimal solution, it must have an optimal solution on the boundary of $\mathcal{P}$.

2. When solving an LP which has multiple optimal solutions, the primal-dual path following algorithm typically converges to an optimal basic feasible solution.

3. Let $\mathcal{P}$ be a polyhedron in the 2-dimensional plane and suppose $z(\mathbf{x})$ is a linear objective function defined on $\mathcal{P}$. If the problem of minimizing $z(\mathbf{x})$ on $\mathcal{P}$ has three distinct extreme points of $\mathcal{P}$ which are optimal, then $z(\mathbf{x})$ is constant on $\mathcal{P}$.

4. The Lagrangean dual provides a no worse bound than the LP relaxation of an integer programming problem. If True, outline the key argument. If False, provide a counter-example.

5. It is possible for the dual of a linear programming problem to have multiple optimal solutions and the primal to have a nondegenerate optimal bfs.

6. Let $\mathbf{c}$ be the cost vector in the objective function of an LP. Let also $\mathbf{x}^*$ be a basic feasible solution (bfs). If for all bases corresponding to $\mathbf{x}^*$ the associated dual basic solution is infeasible, then the optimal objective value must be less than $\mathbf{c}'\mathbf{x}^*$.

7. Consider the problem of finding shortest paths from all nodes of a graph with n nodes to node n. Assume there exists a directed path from each node $1, \ldots, n-1$ to node n and there is no outgoing arc from node n. Then, you can find the shortest paths by solving a network flow problem.

8. Consider the problem of minimizing $\max\{\mathbf{c}'\mathbf{x}, \mathbf{d}'\mathbf{x}\}$, subject to $\mathbf{x} \in \mathcal{P} \subset \mathbb{R}^n$ where

$$\mathcal{P} = \{\mathbf{x} | \mathbf{A}\mathbf{x} \leq \mathbf{b}\}.$$

This problem can be formulated as a linear programming problem. If True, provide the formulation. If False, argue why.

Problem 2 $15 + 10 + 10 + 10 + 15 = 60$ *points*

Consider a zero-sum matrix game with payoff matrix $\mathbf{A} = (a_{ij})_{i=1,\ldots,m}^{j=1,\ldots,n} \in \mathbb{R}^{m\times n}$. The game is played by two players: the "row" player and the "column" player as follows. The row player picks a row $i = 1,\ldots,m$ and the column player picks a column $j = 1,\ldots,n$ resulting in the row player receiving a payoff equal to a_{ij} and the column player receiving a payoff equal to $-a_{ij}$ (hence, a zero-sum game). Both players use *mixed* strategies, that is, the row player selects a probability vector $\mathbf{x} \in \Sigma^m = \{\mathbf{x} = (x_1,\ldots,x_m) \in \mathbb{R}^m \mid \mathbf{x} \geq \mathbf{0}, \sum_{i=1}^m x_i = 1\}$ and the column player selects a a probability vector $\mathbf{y} \in \Sigma^n = \{\mathbf{y} = (y_1,\ldots,y_n) \in \mathbb{R}^n \mid \mathbf{y} \geq \mathbf{0}, \sum_{j=1}^n y_j = 1\}$. We can interpret x_i (respectively, y_j) as the probability that the row player (respectively, the column player) selects row i (respectively, column j). Notice that given strategies $\mathbf{x}$ and $\mathbf{y}$ the expected payoff to the row player is $\mathbf{x}'\mathbf{A}\mathbf{y}$.

(a) Suppose that the row player selects strategy $\mathbf{x}$. Then the best strategy for the column player is

$$\min_{\mathbf{y}\in\Sigma^n} \mathbf{x}'\mathbf{A}\mathbf{y}.$$

What type of problem is this? Formulate it. Show that

$$\min_{\mathbf{y}\in\Sigma^n} \mathbf{x}'\mathbf{A}\mathbf{y} = \min_{j=1,\ldots,n} \left\{ \sum_{i=1}^m x_i a_{ij} \right\}.$$

(The right hand side is the minimum of n numbers.)

(b) Fix $\mathbf{y}$. Show that

$$\max_{\mathbf{x}\in\Sigma^m} \mathbf{x}'\mathbf{A}\mathbf{y} = \max_{i=1,\ldots,m} \left\{ \sum_{j=1}^n y_j a_{ij} \right\}.$$

(c) Show that

$$\max_{\mathbf{x}\in\Sigma^m} \min_{\mathbf{y}\in\Sigma^n} \mathbf{x}'\mathbf{A}\mathbf{y}$$

is equal to the optimal value of the LP

$$\begin{array}{rl} \max & z \\ \text{subject to} & z - \sum_{i=1}^m x_i a_{ij} \le 0, \qquad \forall j = 1,\ldots,n, \\ & \sum_{i=1}^m x_i = 1, \\ & x_i \ge 0, \qquad \forall i = 1,\ldots,m. \end{array}$$

(d) Similarly to part (c) express

$$\min_{\mathbf{y}\in\Sigma^n} \max_{\mathbf{x}\in\Sigma^m} \mathbf{x}'\mathbf{A}\mathbf{y}$$

as the optimal value of some LP.

(e) Show that the order in which the two players select their strategies does not change the expected payoff, i.e.,

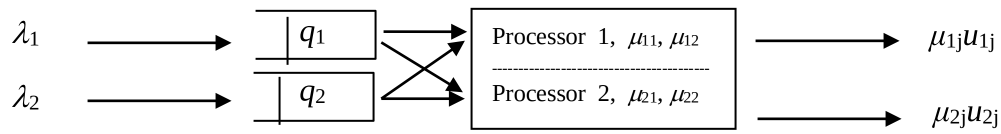
$$\max_{\mathbf{x}\in\Sigma^m} \min_{\mathbf{y}\in\Sigma^n} \mathbf{x}'\mathbf{A}\mathbf{y} = \min_{\mathbf{y}\in\Sigma^n} \max_{\mathbf{x}\in\Sigma^m} \mathbf{x}'\mathbf{A}\mathbf{y}$$

PhD Qualifying Examination, May 2023

Dynamic Programming ID no:______________

Closed Book, Notes, laptop/phone. A 3x5 index card of notes allowed.

Consider a server with two CPUs processing two types of jobs j=1,2 arriving into queue i=1,2 with exponential rate λ_1 and λ_2 respectively. Each of the two processors i=1,2 can process job j with exponential processing time characterized by rate μ_{ij}. The real time dynamic control $u_{ij}(t)$ specifies what type of job j processor i is assigned to process at time t, i.e. $u_{ij}(t)=1$ means that at time t, processor i is assigned to process job j. The allowable control set requires that a single job can be processed by a given processor at a given point in time and that preemption is allowed[a]. If queue i is empty, job type i cannot be processed by any processor. Each queue, q_i, cannot exceed a maximal level of Q_i . If a job of type i arrives while $q_i(t)=Q_i$, the job is turned away (i.e. flow control is exercised) and a cost C_i is incurred. In addition, a cost rate $c_i q_i(t)$ is incurred over time.



1. Derive the Embedded Bellman Equation for the above problem (i.e. define the differential cost function and derive the Bellman Equation as a limit of the finite horizon cost to go function. Make sure to describe the transition probabilities and the enabled events for a given selection of controls.
2. Derive the Uniformized Bellman equation and discuss how the transition probabilities do or do not vary from the case above.
3. Comment briefly on how you can solve the problem above by formulating it as a Linear Programming problem.
4. Describe briefly how the problem above would change if one desired to model the arrival time into queue 1 as a random variable with mean $1/\lambda_1$ and standard coefficient of variation $1/\sqrt{2}$.

[a] This means that at any time a CPU may be reassigned to another queue before it completes processing a job.

SE Qualifying Exam
Nonlinear Systems and Control
May 22, 2023

1. Give a precise mathematical definition of the following terms:

   (a) Uniformly asymptotic stability of an equilibrium point x_0 of $\dot{x}(t) = f(t, x(t))$.

   (b) Positive definiteness of a continuous function $W : [t_0, \infty] \times R^n \to R$.

   (c) Positive limit set of a bounded solution $x(t, x_0, t_0)$.

2. Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -g(x_1)(x_1 + x_2)$$

   where g is locally Lipschitz and $g(y) \geq 1$ for all $y \in R$. Verify that $V(x) = \int_0^{x_1} yg(y)dy + x_1x_2 + x_2^2$ is positive definite for all $x \in R^2$ and radially unbounded, and use it to show that the equilibrium point $x = 0$ is globally asymptotically stable.

3. Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = a \sin x_1 - bu \cos x_1$$

   where a and b are positive constants.

   (a) Show that the system is feedback linearizable.

   (b) Using feedback linearization, design a state feedback controller to stabilize the system at $x_1 = \theta$, where $0 \leq \theta < \pi/2$. Can you make this equilibrium point globally asymptotically stable?